On a conjecture for representations of integers as sums of squares and double shuffle relations

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Abstract

In this paper, we prove a conjecture of Chan and Chua for the number of representations of integers as sums of $8s$ integral squares. The proof uses a theorem of Imamoḡlu and Kohnen and the double shuffle relations satisfied by the double Eisenstein series of level 2.

1 Introduction

For positive integers s and n, let $r_s(n)$ (resp. $t_s(n)$) denote the number of representations of n as a sum of s integral squares (resp. triangular numbers).

$$r_s(n) := \sharp\{(x_1, x_2, \ldots, x_s) \in \mathbb{Z}^s \mid x_1^2 + x_2^2 + \cdots + x_s^2 = n\},$$
$$t_s(n) := \sharp\{(x_1, x_2, \ldots, x_s) \in \mathbb{Z}_{\geq 0}^s \mid x_1(x_1+1)/2 + x_2(x_2+1)/2 + \cdots + x_s(x_s+1)/2 = n\}.$$

Extensive studies on $r_s(n)$ and $t_s(n)$ have been carried out since the times of Fermat, Euler, and Lagrange (see [3, 6]).

Recently, Milne [12, 13, 14] found, then Ono [15], and Rosengren [16], reproved the explicit formulas for $r_{4s^2}(n)$ and $r_{4s^2+4s}(n)$. Kac and Wakimoto [9] found other more complicated, but not as explicit formulas for $r_{4s^2}(n)$ and $r_{4s^2+4s}(n)$. They also conjectured in [9] the explicit formulas for $t_{4s^2}(n)$ and $t_{4s^2+4s}(n)$, as well as a more general formula. Milne [12, 13, 14], then Zagier [18], and Rosengren [17], proved the Kac-Wakimoto conjectured formulas for $t_{4s^2}(n)$ and $t_{4s^2+4s}(n)$, and Zagier [18] independently proved the more general Kac-Wakimoto conjecture. In the current paper, we prove new explicit formulas for $r_{8s}(n)$ and $t_{8s}(n)$ ($s \geq 2$) which were conjectured by Chan and Chua [1] (see also [8, Remark p.820] and [2, 4, 11]).

For integers $k \geq 0$ and $n \geq 1$, we define

$$\sigma_k^{\infty}(n) = \sum_{1 \leq d | n} (-1)^d d^k \text{ and } \sigma_k^0(n) = \sum_{\substack{1 \leq d | n \\ n/d:\text{odd}}} d^k$$

and set $\sigma_k^{\infty}(0) = (1-2^{k+1})B_{k+1}/2(k+1)$ (B_k : Bernoulli number). We also define

$$\rho_{r,s}^{\infty}(n) = \sum_{m=0}^{n} \sigma_r^{\infty}(m)\sigma_s^{\infty}(n-m) \text{ and } \rho_{r,s}^{0}(n) = \sum_{m=1}^{n-1} \sigma_r^{0}(m)\sigma_s^{0}(n-m).$$

Theorem 1. *For any positive integers n and $s \geq 2$, there exist unique rational numbers $\mu_s(l)$ $(l = 2, 3, \ldots, s)$ such that*

$$r_{8s}(n) = (-1)^n \frac{2^{4s}}{(4s-2)!} \sum_{l=2}^{s} \mu_s(l) \binom{4s-2}{2l-1} \rho^{\infty}_{2l-1,4s-2l-1}(n)$$

and

$$t_{8s}(n) = \frac{1}{(4s-2)!} \sum_{l=2}^{s} \mu_s(l) \binom{4s-2}{2l-1} \rho^{0}_{2l-1,4s-2l-1}(n+s).$$

The first several values of $\mu_s(l)$ are given in the following table.

$\mu_s(l)$	$l=2$	$l=3$	$l=4$	$l=5$	$l=6$
$s=2$	36				
$s=3$	420	-200			
$s=4$	3168	-3600	1764		
$s=5$	21060	-30810	36860	-19116	
$s=6$	$\frac{49605048}{343}$	$-\frac{77902500}{343}$	$\frac{15741540}{49}$	$-\frac{139785750}{343}$	$\frac{74727180}{343}$

For the proof of Theorem 1, we use the theory of modular forms on the congruence subgroup $\Gamma_0(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \,\middle|\, c \equiv 0 \pmod{2} \right\}$. Let τ be a variable on the upper half-plane, and

$$\theta(\tau) = \sum_{n \in \mathbb{Z}} q^{n^2}$$

and

$$T(\tau) = q^{1/8} \sum_{n \geq 0} q^{n(n+1)/2}$$

be the standard theta functions ($q = e^{2\pi\sqrt{-1}\tau}$). Then we have, for each positive integer s, $\theta(\tau)^s = \sum_{n\geq 0} r_s(n) q^n$ and $T(\tau)^{8s} = q^s \sum_{n \geq 0} t_{8s}(n) q^n$, and it is well known that the function $\theta(\tau)^s$ is a modular form of weight $s/2$ and level 4 and the function $T(\tau)^{8s}$ is a modular form of weight $4s$ and level 2. The two forms $\theta(\tau)^{8s}$ and $T(\tau)^{8s}$ are related with each other by the transformation formula

$$2^{8s}(2\tau+1)^{-4s} T\left(\frac{-1}{2\tau+1}\right)^{8s} = \theta(\tau)^{8s}.$$

Let $G^0_k(\tau)$ and $G^\infty_k(\tau)$ be the Eisenstein series of weight k on $\Gamma_0(2)$ for the cusps 0 and ∞ (the precise definitions will be recalled in section 2), respectively. Then Theorem 1 is equivalent to the following theorem.

Theorem 2. *For any positive integer $s \geq 2$, there exist unique rational numbers $\mu_s(l)$ $(l = 2, 3, \ldots, s)$ such that*

$$\theta(\tau)^{8s} = 2^{8s} \sum_{l=2}^{s} \mu_s(l) G^\infty_{2l}(\tau + \frac{1}{2}) G^\infty_{4s-2l}(\tau + \frac{1}{2}) \tag{1}$$

and

$$T(\tau)^{8s} = \sum_{l=2}^{s} \mu_s(l) G_{2l}^0(\tau) G_{4s-2l}^0(\tau). \tag{2}$$

In [8], Imamoḡlu and Kohnen proved a similar result stating that $T(\tau)^{8s}$ can be expressed as $\mathbb{Q}$-linear combinations of $G_{2l}^0(\tau)G_{4s-2l}^\infty(\tau)$ $(l = 2, 3, \ldots, 2s-2)$ and $G_{4s}^0(\tau)$ (but the expression is not unique). They proved this by showing that the set $\{G_{2l}^0(\tau)G_{4s-2l}^\infty(\tau)$ $(l = 2, 3, \ldots, 2s-2)\}$ generates the space of cusp forms of weight $4s$ on $\Gamma_0(2)$, using the Rankin–Selberg method and the Eichler–Shimura theory identifying spaces of cusp forms with spaces of periods. We use this result of Imamoḡlu and Kohnen and 'double Eisenstein series' to prove the following theorem, from which Theorem 2 follows.

Theorem 3. *For each positive even integer $k \geq 4$, the set*

$$\{G_k^0(\tau), G_{2l}^0(\tau)G_{k-2l}^0(\tau)\ (2 \leq l \leq [k/4])\}$$

is a basis of the space $\mathbb{Q} \cdot G_k^0(\tau) \oplus S_k^{\mathbb{Q}}(2)$, where $S_k^{\mathbb{Q}}(2)$ is the $\mathbb{Q}$-vector space spanned by cusp forms of weight k on $\Gamma_0(2)$ with rational Fourier coefficients.

In section 2, we will give the proofs of Theorems 2 and 1 assuming Theorem 3. Section 3, which contains a review of the formal double zeta space and the double Eisenstein series, is devoted to the proof of Theorem 3. There we need the double shuffle relations of our newly defined double Eisenstein series whose proof is given in the last section.

2 Proofs of Theorems 1 and 2

In this section, we prove Theorems 1 and 2 assuming Theorem 3. We denote by **ev** (resp. **od**) the set of even integers (resp. odd integers). Let

$$G_k^\infty(\tau) = \frac{1}{2(2\pi\sqrt{-1})^k} \sum_{m \in \mathbf{ev}, n \in \mathbf{od}} \frac{1}{(m\tau + n)^k}$$

and

$$G_k^0(\tau) = \frac{1}{2(2\pi\sqrt{-1})^k} \sum_{m \in \mathbf{od}, n \in \mathbb{Z}} \frac{1}{(m\tau + n)^k}$$

be the Eisenstein series of weight k on $\Gamma_0(2)$ for the cusps ∞ and 0, respectively. For $k > 2$, these series are absolutely convergent, and they have the following Fourier expansions.

$$G_k^\infty(\tau) = (1 - 2^k)\frac{B_k}{2k} + \frac{1}{2^k(k-1)!} \sum_{n>0} \sigma_{k-1}^\infty(n) q^n = \frac{1}{2^k(k-1)!} \sum_{n \geq 0} \sigma_{k-1}^\infty(n) q^n, \tag{3}$$

$$G_k^0(\tau) = \frac{1}{(k-1)!} \sum_{n>0} \sigma_{k-1}^0(n) q^n. \tag{4}$$

Recall that the $\mathbb{Q}$-vector space of modular forms of weight k on $\Gamma_0(2)$ with rational Fourier coefficients is equal to the space $\mathbb{Q} \cdot G_k^0 \oplus \mathbb{Q} \cdot G_k^\infty \oplus S_k^{\mathbb{Q}}(2)$. For each positive integer s, the modular form $T(\tau)^{8s}$ is an element of the space $\mathbb{Q}\cdot G^0_{4s}(\tau)\oplus S^{\mathbb{Q}}_{4s}(2)$ because $\mathrm{ord}_\infty T(\tau)^{8s}>0$, where $\mathrm{ord}_\infty f(\tau)$ is the vanishing order of $f(\tau)$ at the cusp ∞ $(q=0)$. Then, assuming Theorem 3, there exist unique rational numbers $\alpha,\mu_s(l)$ $(l=2,3,\ldots,s)$ such that

$$T(\tau)^{8s}=\alpha G^0_{4s}(\tau)+\sum_{l=2}^{s}\mu_s(l)G^0_{2l}(\tau)G^0_{4s-2l}(\tau).$$

Since $\mathrm{ord}_\infty T(\tau)^{8s}=s\geq 2, \mathrm{ord}_\infty G^0_{4s}(\tau)=1$ and $\mathrm{ord}_\infty G^0_{2l}(\tau)G^0_{4s-2l}(\tau)=2$ $(2\leq l\leq s)$, we find that $\alpha=0$. Thus, we have the assertion (2) of Theorem 2. On the other hand, from the transformation formulas

$$2^{8s}(2\tau)^{-4s}T(\frac{-1}{2\tau})^{8s}=\theta(\tau+\frac{1}{2})^{8s}\quad(s\geq 1)$$

and

$$(2\tau)^{-k}G^0_k(\frac{-1}{2\tau})=G^\infty_k(\tau)\quad(k\geq 4:\text{even}),$$

we have

$$\begin{aligned}\theta(\tau+\frac{1}{2})^{8s}&=2^{8s}(2\tau)^{-4s}T(-\frac{1}{2\tau})^{8s}\\&=2^{8s}(2\tau)^{-4s}\sum_{l=2}^{s}\mu_s(l)G^0_{2l}(-\frac{1}{2\tau})G^0_{4s-2l}(-\frac{1}{2\tau})\\&=2^{8s}\sum_{l=2}^{s}\mu_s(l)G^\infty_{2l}(\tau)G^\infty_{4s-2l}(\tau).\end{aligned}$$

Hence, we have the formula (1) by letting $\tau\to\tau+1/2$. This completes the proof of Theorem 2. Consequently, we have

$$\begin{aligned}T(\tau)^{8s}&=q^s\sum_{n\geq 0}t_{8s}(n)q^n\\&=\sum_{n>0}\left(\sum_{l=2}^{s}\frac{\mu_s(l)}{(2l-1)!(4s-2l-1)!}\sum_{m=1}^{n-1}\sigma^0_{2l-1}(m)\sigma^0_{4s-2l-1}(n-m)\right)q^n\\&=\frac{1}{(4s-2)!}\sum_{n>0}\left(\sum_{l=2}^{s}\mu_s(l)\binom{4s-2}{2l-1}\rho^0_{2l-1,4s-2l-1}(n)\right)q^n\end{aligned}$$

and

$$\begin{aligned}\theta(\tau)^{8s}&=\sum_{n\geq 0}r_{8s}(n)q^n\\&=2^{8s}\sum_{n\geq 0}(-1)^n\left(\sum_{l=2}^{s}\frac{\mu_s(l)}{2^{4s}(2l-1)!(4s-2l-1)!}\sum_{m=0}^{n}\sigma^\infty_{2l-1}(m)\sigma^\infty_{4s-2l-1}(n-m)\right)q^n\\&=\frac{2^{4s}}{(4s-2)!}\sum_{n\geq 0}(-1)^n\left(\sum_{l=2}^{s}\mu_s(l)\binom{4s-2}{2l-1}\rho^\infty_{2l-1,4s-2l-1}(n)\right)q^n,\end{aligned}$$

which give Theorem 1 by comparing the coefficients of q^n. $\square$

3 The double Eisenstein series and the proof of Theorem 3

Let $G_k(\tau)$ be the Eisenstein series on $\mathrm{SL}_2(\mathbb{Z})$ (we will give the definition below). We first give a brief exposition of the strategy of our proof. We assume $k \geq 4$ is even. We want to find $\mathbb{Q}$-linear relations expressing $G^0_{2l}(\tau)G^\infty_{k-2l}(\tau)$ $(2 \leq l \leq k/2-2)$, which are the Imamoḡlu-Kohnen generators of $S^{\mathbb{Q}}_k(2)$, as sums of $G^0_k(\tau)$ and $G^0_{2l}(\tau)G^0_{k-2l}(\tau)$ $(2 \leq l \leq [k/4])$. Since it holds

$$G^0_{2l}(\tau)G^\infty_{k-2l}(\tau) = 2^{-k+2l}((2^{k-2l}-1)G^0_{2l}(\tau)G_{k-2l}(2\tau) - G^0_{2l}(\tau)G^0_{k-2l}(\tau))$$

for $l = 2, 3, \ldots, k/2-2$ (we will show this below), we will show that all $G^0_{2l}(\tau)G_{k-2l}(2\tau)$ $(2 \leq l \leq k/2-2)$ can be expressed as sums of $G^0_k(\tau)$ and $G^0_{2l}(\tau)G^0_{k-2l}(\tau)$ $(2 \leq l \leq [k/4])$. Then, since $\dim S^{\mathbb{Q}}_k(2) = [k/4]-1$, we find that the set $\{G^0_k(\tau), G^0_{2l}(\tau)G^0_{k-2l}(\tau) \mid 2 \leq l \leq [k/4]\}$ forms a basis of the space $\mathbb{Q}\cdot G^0_k(\tau) \oplus S^{\mathbb{Q}}_k(2)$.

Such relations are obtained by using the same method of the previous work of Kaneko and the author [10] (in the proof of Theorem 5 of [10] they proved that all $G^0_{2l}(\tau)G^\infty_{k-2l}(\tau)$ $(2 \leq l \leq k/2-2)$ can be written as $\mathbb{Q}$-linear combinations of $G^\infty_k(\tau)$ and $G^\infty_{2l}(\tau)G^\infty_{k-2l}(\tau)$ $(2 \leq l \leq [k/4])$). Now we mention a basic idea inspired by [5]. For seeking the relations among such modular forms, we consider the decomposition of two products of the Eisenstein series into two expressions as sums of certain q-series which we call double Eisenstein series. (For the original version we refer the reader to [5, Section 7].) In order to do this, we consider the function

$$\sum_{\substack{m\tau+n\succ 0\\ m,n\in\mathbb{Z}}} \frac{1}{(m\tau+n)^k},$$

where the inequality $m\tau + n \succ 0$ means $m > 0$ or $m = 0, n > 0$. When k is even and $k \geq 4$, this is just the half of the usual Eisenstein series of weight k on $\mathrm{SL}_2(\mathbb{Z})$, and when k is odd≥ 3, this defines a non-zero periodic function. By the definition, for $k > 2$, usual computations show that

$$\sum_{\substack{m\tau+n\succ 0\\ m,n\in\mathbb{Z}}} \frac{1}{(m\tau+n)^k} = \sum_{n>0}\frac{1}{n^k} + \sum_{\substack{m>0\\ n\in\mathbb{Z}}} \frac{1}{(m\tau+n)^k} = \zeta(k) + \frac{(-2\pi\sqrt{-1})^k}{(k-1)!}\sum_{n>0}\sigma_{k-1}(n)q^n,$$

where $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$. We can obtain the following decomposition of product, and the double Eisenstein series appears on the right-hand side:

$$\sum_{\substack{m\tau+n\succ 0\\ m,n\in\mathbb{Z}}} \frac{1}{(m\tau+n)^r} \sum_{\substack{m'\tau+n'\succ 0\\ m',n'\in\mathbb{Z}}} \frac{1}{(m'\tau+n')^s} =$$

$$\left(\sum_{\substack{m\tau+n\succ m'\tau+n'\succ 0\\ m,n,m',n'\in\mathbb{Z}}} + \sum_{\substack{m'\tau+n'\succ m\tau+n\succ 0\\ m,n,m',n'\in\mathbb{Z}}} + \sum_{\substack{m\tau+n=m'\tau+n'\succ 0\\ m,n,m',n'\in\mathbb{Z}}}\right) \frac{1}{(m\tau+n)^r(m'\tau+n')^s},$$

where $m\tau + n \succ m'\tau + n'$ means $(m-m')\tau + (n-n') \succ 0$. This decomposition is usually called the harmonic (or stuffle) product. Another decomposition, which is called the shuffle product, is obtained by putting $X = m\tau+n, Y = m'\tau+n'$ and summing all positive elements on $\mathbb{Z}\tau + \mathbb{Z}$ for the following partial fraction decomposition (see [5, eq. (19)])

$$\frac{1}{X^r Y^s} = \sum_{\substack{i+j=r+s\\ i,j\geq 1}} \left(\frac{\binom{i-1}{r-1}}{(X+Y)^i Y^j} + \frac{\binom{i-1}{s-1}}{(X+Y)^i X^j} \right).$$

These two expansions of two products of the Eisenstein series give $\mathbb{Q}$-linear relations among double Eisenstein series, which we call double shuffle relations. (In the study of multiple zeta values, the double shuffle relations are very important (see [7]).) Considering combinations of the double shuffle relations of double Eisenstein series, we can obtain non-trivial relations among modular forms. For example, combining the double shuffle relations of weight 12, we have

$$84G_4(\tau)G_8(\tau) + 50G_6(\tau)^2 = 143G_{12}(\tau).$$

Here we note that the double shuffle relations as mentioned above apparently hold only in the case $r, s > 2$, where the defining series are absolutely convergent. But it is known that these relations can be extended to all integer $r, s \geq 1$ with $(r, s) \neq (1, 1)$ by extending the definition of the double Eisenstein series as Fourier series adding a certain correction term ‘ $\varepsilon_{r,s}(\tau)$ ’, which is 0 for $r \geq 3$ and $s \geq 2$. (We carry out this extension in Proposition 6 and then Definition 7.)

Now we state our double shuffle relations. We define the holomolphic function $G_k(\tau)$ on the upper half-plane by

$$G_k(\tau) = \widetilde{\zeta}(k) + \frac{(-1)^k}{(k-1)!} \sum_{n>0} \sigma_{k-1}(n) q^n \quad (k \geq 1),$$

where $\widetilde{\zeta}(k) = (2\pi\sqrt{-1})^{-k}\zeta(k)$ $(k \geq 2)$ or 0 $(k = 1)$. For $k \geq 1$, we let

$$G_k^\infty(\tau) = G_k(2\tau) - 2^{-k}G_k(\tau), \tag{5}$$
$$G_k^0(\tau) = G_k(\tau) - G_k(2\tau), \tag{6}$$

which coincide with (3) and (4) when $k \geq 4$ is even. The series $G_k(\tau)$ and $G_k^0(\tau)$ are the Fourier series of

$$G_k(\tau) = \frac{1}{(2\pi\sqrt{-1})^k} \sum_{\substack{m\tau+n\succ 0\\ m,n\in\mathbb{Z}}} \frac{1}{(m\tau+n)^k} \tag{7}$$

and

$$G_k^0(\tau) = \frac{1}{(2\pi\sqrt{-1})^k} \sum_{\substack{m\tau+n\succ 0\\ m\in\mathbf{od},n\in\mathbb{Z}}} \frac{1}{(m\tau+n)^k}, \tag{8}$$

respectively, when $k > 2$. For positive integers r and s, we put

$$P_{r,s}^{\mathbf{oe}}(\tau) = G_r^0(\tau)G_s(2\tau) + \delta_{r,2}\frac{G_s'(2\tau)}{4s} + \delta_{s,2}\frac{G_r^0(\tau)'}{4r} \tag{9}$$

and

$$P_{r,s}^{\mathbf{oo}}(\tau) = G_r^0(\tau)G_s^0(\tau) + \delta_{r,2}\frac{G_s^0(\tau)'}{4s} + \delta_{s,2}\frac{G_r^0(\tau)'}{4r}, \tag{10}$$

where the differential operator $'$ means $q\cdot d/dq$ and $\delta_{r,s}$ is the Kronecker delta. It can be shown that both $P^{\mathbf{oe}}_{r,s}(\tau)$ and $P^{\mathbf{oo}}_{r,s}(\tau)$ are modular forms on $\Gamma_0(2)$ when both r and s are even greater than 1. For the harmonic and the shuffle product of each $P^{\mathbf{oe}}_{r,s}(\tau)$ and $P^{\mathbf{oo}}_{r,s}(\tau)$, the following double Eisenstein series are needed:

$$Z^{\mathbf{eo}}_{r,s}(\tau)=\frac{1}{(2\pi\sqrt{-1})^{r+s}}\sum_{\substack{\lambda\succ\mu\succ0\\ \lambda\in\mathbf{ev}\cdot\tau+\mathbb{Z}\\ \mu\in\mathbf{od}\cdot\tau+\mathbb{Z}}}\frac{1}{\lambda^r\mu^s},\ Z^{\mathbf{oe}}_{r,s}(\tau)=\frac{1}{(2\pi\sqrt{-1})^{r+s}}\sum_{\substack{\lambda\succ\mu\succ0\\ \lambda\in\mathbf{od}\cdot\tau+\mathbb{Z}\\ \mu\in\mathbf{ev}\cdot\tau+\mathbb{Z}}}\frac{1}{\lambda^r\mu^s},$$

$$Z^{\mathbf{oo}}_{r,s}(\tau)=\frac{1}{(2\pi\sqrt{-1})^{r+s}}\sum_{\substack{\lambda\succ\mu\succ0\\ \lambda\in\mathbf{od}\cdot\tau+\mathbb{Z}\\ \mu\in\mathbf{od}\cdot\tau+\mathbb{Z}}}\frac{1}{\lambda^r\mu^s}\tag{11}$$

for $r\geq 3$ and $s\geq 2$, which ensure the absolute convergence of the defining series (11). In Section 4, we extend the definition of the double Eisenstein series in (11) to all integers $r,s\geq 1$ as q-series, which by Proposition 6, are the Fourier expansions of (11) for $r\geq 3$ and $s\geq 2$. We then prove the following double shuffle relations:

Proposition 4. *For positive even integer k and all integers $r,s\geq 1$ with $r+s=k$, one has*

$$P^{\mathbf{oe}}_{r,s}(\tau)=Z^{\mathbf{oe}}_{r,s}(\tau)+Z^{\mathbf{eo}}_{s,r}(\tau)=\sum_{\substack{i+j=k\\ i,j\geq1}}\binom{i-1}{r-1}Z^{\mathbf{oe}}_{i,j}(\tau)+\sum_{\substack{i+j=k\\ i,j\geq1}}\binom{i-1}{s-1}Z^{\mathbf{oo}}_{i,j}(\tau),\tag{12}$$

$$P^{\mathbf{oo}}_{r,s}(\tau)=Z^{\mathbf{oo}}_{r,s}(\tau)+Z^{\mathbf{oo}}_{s,r}(\tau)+G^0_k(\tau)=\sum_{\substack{i+j=k\\ i,j\geq1}}\left(\binom{i-1}{r-1}+\binom{i-1}{s-1}\right)Z^{\mathbf{eo}}_{i,j}(\tau).\tag{13}$$

Note that from expressions (7), (8) and (11), and using the harmonic and the shuffle product, we can verify Proposition 4 for $r,s\geq 3$. The point is to extend these to the case of non-absolute convergence.

Assuming Proposition 4, we prove Theorem 3 using a theorem of Kaneko and the author [10, Theorem 1]. To state Theorem 1 of [10], we define the formal double zeta space for level 2 as the $\mathbb{Q}$-vector space generated by formal variables $Z^{\mathbf{eo}}_{r,s},Z^{\mathbf{oe}}_{r,s},Z^{\mathbf{oo}}_{r,s},P^{\mathbf{oe}}_{r,s},P^{\mathbf{oo}}_{r,s}$ $(r+s=k,r,s\geq 1)$ and $Z^{\mathbf{o}}_k$ with relations

$$P^{\mathbf{oe}}_{r,s}=Z^{\mathbf{oe}}_{r,s}+Z^{\mathbf{eo}}_{s,r}=\sum_{\substack{i+j=k\\ i,j\geq1}}\binom{i-1}{r-1}Z^{\mathbf{oe}}_{i,j}+\sum_{\substack{i+j=k\\ i,j\geq1}}\binom{i-1}{s-1}Z^{\mathbf{oo}}_{i,j},$$

$$P^{\mathbf{oo}}_{r,s}=Z^{\mathbf{oo}}_{r,s}+Z^{\mathbf{oo}}_{s,r}+Z^{\mathbf{o}}_k=\sum_{\substack{i+j=k\\ i,j\geq1}}\left(\binom{i-1}{r-1}+\binom{i-1}{s-1}\right)Z^{\mathbf{eo}}_{i,j}.\tag{14}$$

We denote this space by $\mathcal{D}_k$. Note that from Proposition 4, there is the surjection map from $\mathcal{D}_k$ to the space spanned by the double Eisenstein series of weight $(=r+s)$ k which sends each generator as follows:

$$P^{\mathbf{oe}}_{r,s}\mapsto P^{\mathbf{oe}}_{r,s}(\tau),\ P^{\mathbf{oo}}_{r,s}\mapsto P^{\mathbf{oo}}_{r,s}(\tau),\ Z^{\mathbf{o}}_k\mapsto G^0_k(\tau),$$
$$Z^{\mathbf{eo}}_{r,s}\mapsto Z^{\mathbf{eo}}_{r,s}(\tau),\ Z^{\mathbf{oe}}_{r,s}\mapsto Z^{\mathbf{oe}}_{r,s}(\tau),\ Z^{\mathbf{oo}}_{r,s}\mapsto Z^{\mathbf{oo}}_{r,s}(\tau).\tag{15}$$

For the space $\mathcal{D}_k$, they showed

Theorem 5. ([10, Theorem 1]) *Suppose k is even and $k \geq 4$. In $\mathcal{D}_k$, we have*

1)

$$\sum_{\substack{r=2\\ r:\text{even}}}^{k-2} Z^{\mathbf{oo}}_{r,k-r} = \frac{1}{4} Z^{\mathbf{o}}_k. \tag{16}$$

2) Each $P^{\mathbf{oe}}_{r,k-r}$ with r even can be written as a $\mathbb{Q}$-linear combination of $P^{\mathbf{oo}}_{i,j}$ $(i,j:$ even$, i+j=k)$ and $Z^{\mathbf{o}}_k$.

Using the first equality of (14) and (16), we have

$$(1 + 4 \cdot [k/4]) Z^{\mathbf{o}}_k = 2 \sum_{r=1}^{[k/4]} (2 - \delta_{r,k/4}) P^{\mathbf{oo}}_{2r,k-2r},$$

where $\delta_{i,j}$ is the Kronecker delta. This implies that $P^{\mathbf{oo}}_{2,k-2}$ can be written as $\mathbb{Q}$-linear combinations of $P^{\mathbf{oo}}_{2r,k-2r}$ $(2 \leq r \leq [k/4])$ and $Z^{\mathbf{o}}_k$. Therefore the $\mathbb{Q}$-vector subspace of $\mathcal{D}_k$ spanned by $P^{\mathbf{oe}}_{r,s}, P^{\mathbf{oo}}_{r,s}$ $(r,s:\text{even})$ and $Z^{\mathbf{o}}_k$ with $r+s=k$ is generated by $P^{\mathbf{oo}}_{2r,k-2r}$ $(2 \leq r \leq [k/4])$ and $Z^{\mathbf{o}}_k$. Summarizing, for even $k \geq 4$, Theorem 5 says that

$$\langle P^{\mathbf{oe}}_{2r,k-2r}, P^{\mathbf{oo}}_{2r,k-2r}, Z^{\mathbf{o}}_k \mid 1 \leq r \leq k/2-1 \rangle_{\mathbb{Q}} = \langle P^{\mathbf{oo}}_{2r,k-2r}, Z^{\mathbf{o}}_k \mid 2 \leq r \leq [k/4] \rangle_{\mathbb{Q}}. \tag{17}$$

Finally, we prove Theorem 3.

Proof. By (5) and (6), one has

$$G^0_r(\tau) G^\infty_s(\tau) = 2^{-s}((2^s-1)P^{\mathbf{oe}}_{r,s}(\tau) - P^{\mathbf{oo}}_{r,s}(\tau)) \quad (r,s \geq 4, \text{which implies } \delta_{r,2} = \delta_{s,2} = 0). \tag{18}$$

As mentioned in section 1, the products $G^0_{2l}(\tau) G^\infty_{k-2l}(\tau)$ $(l = 2,3,\ldots,k/2-2)$ generate $S^{\mathbb{Q}}_k(2)$. Therefore, by (17), (18) and (15), the space spanned by $P^{\mathbf{oo}}_{2r,k-2r}(\tau)$ $(2 \leq r \leq [k/4])$ and $G^0_k(\tau)$ contains $\mathbb{Q} \cdot G^0_k(\tau) \oplus S^{\mathbb{Q}}_k(2)$. The number of generators of the space $\langle P^{\mathbf{oo}}_{2r,k-2r}(\tau), G^0_k(\tau) \mid 2 \leq r \leq [k/4] \rangle_{\mathbb{Q}}$ is equal to $[k/4]$, and hence we have

$$\langle P^{\mathbf{oo}}_{2r,k-2r}(\tau), G^0_k(\tau) \mid 2 \leq r \leq [k/4] \rangle_{\mathbb{Q}} = \mathbb{Q} \cdot G^0_k(\tau) \oplus S^{\mathbb{Q}}_k(2).$$

This completes the proof of Theorem 3. $\square$

Remark. The fact that the Fricke involution $W_2 = \left(\begin{smallmatrix} 0 & 1 \\ -2 & 0 \end{smallmatrix}\right)$ transforms $G^0_k(\tau)$ into $G^\infty_k(\tau)$ reduces Theorem 3 to the previous result of M. Kaneko and the author [10, Theorem 5 (2)]. (This remark was given by Dr. Shuji Yamamoto and Dr. Toshiyuki Kikuta, and the author would like to thank them for useful comments.) However, we think the double shuffle relations in the present case are of independent interest, we adapt our original proof here.

4 Double Eisenstein series and double shuffle relations

In this section, we give an extended definition of the double Eisenstein series and prove their double shuffle relations (Proposition 4). Since the double Eisenstein series defined by (11) is invariant under the translation $\tau \to \tau+1$, we first compute the Fourier expansion of (11) for

$r \geq 3$ and $s \geq 2$. Starting with these identities, and adding a suitable correction term, we extend the definition of the double Eisenstein series in (11) to all integers $r, s \geq 1$ as q-series. To describe the Fourier expansions, we use the power series $\varphi_k(\tau)$ in $\mathbb{Q}[[q]]$ defined by

$$\varphi_k(\tau) = \frac{(-1)^k}{(k-1)!} \sum_{u>0} u^{k-1} q^u \quad (k \geq 1).$$

Proposition 6. *For any integers $r \geq 3$ and $s \geq 2$, we have*

$$\begin{aligned}
Z^{\mathbf{eo}}_{r,s}(\tau) &= \sum_{\substack{m>m'>0 \\ m\in\mathbf{ev}, m'\in\mathbf{od}}} \varphi_r(m\tau)\varphi_s(m'\tau), \\
Z^{\mathbf{oe}}_{r,s}(\tau) &= \sum_{\substack{m>m'>0 \\ m\in\mathbf{od}, m'\in\mathbf{ev}}} \varphi_r(m\tau)\varphi_s(m'\tau) + \widetilde{\zeta}(s) \sum_{m\in\mathbf{od}_{>0}} \varphi_r(m\tau), \\
Z^{\mathbf{oo}}_{r,s}(\tau) &= \sum_{\substack{m>m'>0 \\ m\in\mathbf{od}, m'\in\mathbf{od}}} \varphi_r(m\tau)\varphi_s(m'\tau) \\
&\quad + \sum_{\substack{p+h=r+s \\ p,h\geq 1}} \left((-1)^s \binom{p-1}{s-1} + (-1)^{p+r} \binom{p-1}{r-1} \right) \widetilde{\zeta}(p) \sum_{m\in\mathbf{od}_{>0}} \varphi_h(m\tau).
\end{aligned}$$

Proof. We first recall the Lipschitz formula

$$\begin{aligned}
\lim_{N\to\infty} \sum_{n=-N}^{N} \frac{1}{\tau+n} &= -\pi i + (-2\pi\sqrt{-1}) \sum_{u>0} q^u = -\pi i + 2\pi\sqrt{-1}\varphi_1(\tau), \\
\sum_{n\in\mathbb{Z}} \frac{1}{(\tau+n)^r} &= \frac{(-2\pi\sqrt{-1})^r}{(r-1)!} \sum_{u>0} u^{r-1} q^u = (2\pi\sqrt{-1})^r \varphi_r(\tau) \quad (r \geq 2).
\end{aligned}$$

We can divide the summation in the defining series (11) into four terms, corresponding to $m = m' = 0$, $m > m' = 0$, $m = m' > 0$, and $m > m' > 0$, where the first term is zero for the cases $Z^{\mathbf{eo}}_{r,s}, Z^{\mathbf{oe}}_{r,s}$, and $Z^{\mathbf{oo}}_{r,s}$ (for convenience, we sometimes drop the variable τ). We only prove the identity involving $Z^{\mathbf{oo}}_{r,s}$. In this case, we obtain

$$Z^{\mathbf{oo}}_{r,s} = \frac{1}{(2\pi\sqrt{-1})^{r+s}} \Big(\sum_{\substack{m=m'>0 \\ n>n' \\ m,m'\in\mathbf{od} \\ n,n'\in\mathbb{Z}}} + \sum_{\substack{m>m'>0 \\ m,m'\in\mathbf{od} \\ n,n'\in\mathbb{Z}}} \Big) \frac{1}{(m\tau+n)^r(m'\tau+n')^s}.$$

The second term is easily seen to be

$$\sum_{\substack{m>m'>0 \\ m,m'\in\mathbf{od}}} \varphi_r(m\tau)\varphi_s(m'\tau).$$

For the calculation of the first term, we need the partial fraction decomposition

$$\begin{aligned}\frac{1}{(\tau+n)^r(\tau+n')^s} =&(-1)^s\sum_{i=0}^{r-1}\binom{s+i-1}{i}\frac{1}{(\tau+n)^{r-i}}\cdot\frac{1}{(n-n')^{s+i}}\\ &+\sum_{j=0}^{s-1}(-1)^j\binom{r+j-1}{j}\frac{1}{(\tau+n')^{s-j}}\cdot\frac{1}{(n-n')^{r+j}}.\end{aligned} \tag{19}$$

Let $h=n-n'$. Then h is a positive integer. Using (19), the first term can be calculated as

$$\begin{aligned}&\frac{1}{(2\pi\sqrt{-1})^{r+s}}\sum_{\substack{m=m'>0\\ n>n'\\ m,m'\in\mathbf{od},n,n'\in\mathbb{Z}}}\frac{1}{(m\tau+n)^r(m'\tau+n')^s}=\frac{1}{(2\pi\sqrt{-1})^{r+s}}\sum_{m\in\mathbf{od}_{>0}}\sum_{\substack{n>n'\\ n,n'\in\mathbb{Z}}}\frac{1}{(m\tau+n)^r(m\tau+n')^s}\\
=&\frac{1}{(2\pi\sqrt{-1})^{r+s}}\sum_{m\in\mathbf{od}_{>0}}\sum_{\substack{n\in\mathbb{Z}\\ h\in\mathbb{Z}_{>0}}}\left\{(-1)^s\sum_{i=0}^{r-1}\binom{s+i-1}{i}\frac{1}{(m\tau+n)^{r-i}}\frac{1}{h^{s+i}}\right.\\
&\left.+\sum_{j=0}^{s-1}(-1)^j\binom{r+j-1}{j}\frac{1}{(m\tau+n-h)^{s-j}}\frac{1}{h^{r+j}}\right\}\\
=&(-1)^s\sum_{i=0}^{r-1}\binom{s+i-1}{i}\sum_{h\in\mathbb{Z}_{>0}}\frac{(2\pi\sqrt{-1})^{-s-i}}{h^{s+i}}\sum_{m\in\mathbf{od}_{>0}}\sum_{n\in\mathbb{Z}}\frac{(2\pi\sqrt{-1})^{-r+i}}{(m\tau+n)^{r-i}}\\
&+\sum_{j=0}^{s-1}(-1)^j\binom{r+j-1}{j}\sum_{h\in\mathbb{Z}_{>0}}\frac{(2\pi\sqrt{-1})^{-r-j}}{h^{r+j}}\sum_{m\in\mathbf{od}_{>0}}\sum_{n\in\mathbb{Z}}\frac{(2\pi\sqrt{-1})^{-s+j}}{(m\tau+n-h)^{s-j}}\\
=&(-1)^s\sum_{i=0}^{r-2}\binom{s+i-1}{i}\widetilde{\zeta}(s+i)\sum_{m\in\mathbf{od}_{>0}}\varphi_{r-i}(m\tau)\\
&+\sum_{j=0}^{s-2}(-1)^j\binom{r+j-1}{j}\widetilde{\zeta}(r+j)\sum_{m\in\mathbf{od}_{>0}}\varphi_{s-j}(m\tau)\\
=&\sum_{\substack{p+h=r+s\\ p,h\geq 1}}\left\{(-1)^s\binom{p-1}{s-1}+(-1)^{p+r}\binom{p-1}{r-1}\right\}\widetilde{\zeta}(p)\sum_{m\in\mathbf{od}_{>0}}\varphi_h(m\tau).\end{aligned}$$

The cancellation of the terms for $i=r-1$ and $j=s-1$ in the third equality can be justified by computing Cauchy principal values. The final equality is obtained by setting $s+i=p, r-i=h$ in the first term and $r+j=p, s-j=h$ in the second. This completes the proof for $Z^{\mathbf{oo}}_{r,s}$, the verification of the other cases being left to the reader. □

For a positive integer r and non-negative integer s, we put

$$\begin{aligned}&f^{\mathbf{o}}_r(\tau)=\sum_{m\in\mathbf{od}_{>0}}\varphi_r(m\tau),\ f^{\mathbf{e}}_r(\tau)=\sum_{m\in\mathbf{ev}_{>0}}\varphi_r(m\tau),\\ &\overline{f}^{\mathbf{o}}_s(\tau)=-\sum_{m\in\mathbf{od}_{>0}}m\varphi_{s+1}(m\tau),\ \overline{f}^{\mathbf{e}}_s(\tau)=-\sum_{m\in\mathbf{ev}_{>0}}m\varphi_{s+1}(m\tau).\end{aligned}$$

Since the functions G_k and G_k^0 ($k \geq 1$) can be written in the forms

$$G_k(\tau) = \widetilde{\zeta}(k) + \frac{(-1)^k}{(k-1)!} \sum_{\substack{u>0 \\ m>0}} u^{k-1} q^{um} \quad \text{and} \quad G_k^0(\tau) = \frac{(-1)^k}{(k-1)!} \sum_{\substack{u>0 \\ m \in \mathbf{od}_{>0}}} u^{k-1} q^{um},$$

respectively, one finds

$$G_k^0(\tau) = f_k^{\mathbf{o}}(\tau), G_k(2\tau) = \widetilde{\zeta}(k) + f_k^{\mathbf{e}}(\tau), G_k^0(\tau)' = k\overline{f}_k^{\mathbf{o}}(\tau), G_k'(2\tau) = k\overline{f}_k^{\mathbf{e}}(\tau).$$

Then, for $r, s \geq 1$, the functions $P_{r,s}^{\mathbf{oe}}(\tau)$ and $P_{r,s}^{\mathbf{oo}}(\tau)$ defined in (9), (10) can be written in the forms

$$P_{r,s}^{\mathbf{oe}}(\tau) = f_r^{\mathbf{o}}(\tau)(\widetilde{\zeta}(s) + f_s^{\mathbf{e}}(\tau)) + \frac{1}{4}(\delta_{r,2}\overline{f}_s^{\mathbf{e}}(\tau) + \delta_{s,2}\overline{f}_r^{\mathbf{o}}(\tau)),$$
$$P_{r,s}^{\mathbf{oo}}(\tau) = f_r^{\mathbf{o}}(\tau)f_s^{\mathbf{o}}(\tau) + \frac{1}{4}(\delta_{r,2}\overline{f}_s^{\mathbf{o}}(\tau) + \delta_{s,2}\overline{f}_r^{\mathbf{o}}(\tau)).$$

To extend the definition of the double Eisenstein series $Z_{r,s}^{\mathbf{eo}}, Z_{r,s}^{\mathbf{oe}}$, and $Z_{r,s}^{\mathbf{oo}}$ in (11) to any positive integers r and s, we need the correction terms $\varepsilon_{r,s}^{\mathbf{eo}}(\tau), \varepsilon_{r,s}^{\mathbf{oe}}(\tau)$ and $\varepsilon_{r,s}^{\mathbf{oo}}(\tau)$. This is necessary because we require the extended double Eisenstein series to satisfy the double shuffle relations (see [5, 10]). We set

$$\varepsilon_{r,s}^{\mathbf{eo}}(\tau) = \delta_{r,2}\overline{f}_s^{\mathbf{o}}(\tau) - \delta_{r,1}\overline{f}_{s-1}^{\mathbf{o}}(\tau) + \delta_{s,1}\overline{f}_{r-1}^{\mathbf{e}}(\tau) + \delta_{r,1}\delta_{s,1}\alpha_1,$$
$$\varepsilon_{r,s}^{\mathbf{oe}}(\tau) = \delta_{r,2}\overline{f}_s^{\mathbf{e}}(\tau) - \delta_{r,1}\overline{f}_{s-1}^{\mathbf{e}}(\tau) + \delta_{s,1}\overline{f}_{r-1}^{\mathbf{o}}(\tau) + \delta_{r,1}\delta_{s,1}\alpha_2,$$
$$\varepsilon_{r,s}^{\mathbf{oo}}(\tau) = \delta_{r,2}\overline{f}_s^{\mathbf{o}}(\tau) - \delta_{r,1}\overline{f}_{s-1}^{\mathbf{o}}(\tau) + \delta_{s,1}(\overline{f}_{r-1}^{\mathbf{o}}(\tau) + 2f_r^{\mathbf{o}}(\tau)) + \delta_{r,1}\delta_{s,1}\alpha_3,$$

where $\alpha_1 = -\alpha_2 = \overline{f}_0^{\mathbf{o}}(\tau)$ and $\alpha_3 = 2\overline{f}_0^{\mathbf{o}}(\tau) + \overline{f}_0^{\mathbf{e}}(\tau)$.

Definition 7. *For positive integers r and s, we now define*

$$Z_{r,s}^{\mathbf{eo}}(\tau) = \sum_{\substack{m>m'>0 \\ m \in \mathbf{ev}, m' \in \mathbf{od}}} \varphi_r(m\tau)\varphi_s(m'\tau) + \frac{1}{4}\varepsilon_{r,s}^{\mathbf{eo}}(\tau),$$
$$Z_{r,s}^{\mathbf{oe}}(\tau) = \sum_{\substack{m>m'>0 \\ m \in \mathbf{od}, m' \in \mathbf{ev}}} \varphi_r(m\tau)\varphi_s(m'\tau) + \widetilde{\zeta}(s)f_r^{\mathbf{o}}(\tau) + \frac{1}{4}\varepsilon_{r,s}^{\mathbf{oe}}(\tau),$$
$$Z_{r,s}^{\mathbf{oo}}(\tau) = \sum_{\substack{m>m'>0 \\ m \in \mathbf{od}, m' \in \mathbf{od}}} \varphi_r(m\tau)\varphi_s(m'\tau)$$
$$+ \sum_{\substack{p+h=r+s \\ p,h \geq 1}} \left((-1)^s \binom{p-1}{s-1} + (-1)^{p+r} \binom{p-1}{r-1}\right) \widetilde{\zeta}(p) f_h^{\mathbf{o}}(\tau) + \frac{1}{4}\varepsilon_{r,s}^{\mathbf{oo}}(\tau).$$

We now prove Proposition 4.

Proof. Let $k = r + s$. The proof will be divided into two steps. We first prove the equalities of the imaginary parts in Proposition 4. The only imaginary parts that appear come from the constant terms $\widetilde{\zeta}(s)$ of $G_s(2\tau)$ (s : odd), $\widetilde{\zeta}(s)$ in $Z^{\mathbf{oe}}_{r,s}(\tau)$ (s : odd) or $\widetilde{\zeta}(p)$ (p : odd) in $Z^{\mathbf{oo}}_{r,s}(\tau)$. In (13), no imaginary part appears since those of $Z^{\mathbf{oo}}_{r,s}(\tau)+Z^{\mathbf{oo}}_{s,r}(\tau)$ cancels. To prove the equalities of the imaginary parts of (12), we consider the generating functions as follows:

$$
\begin{aligned}
Z^{\mathbf{oo}}_k(X,Y) &:= \sum_{\substack{r+s=k\\ r,s\geq 1}} \mathrm{Im}\ Z^{\mathbf{oo}}_{r,s} X^{r-1}Y^{s-1}\\
&= \sum_{\substack{r+s=k\\ r,s\geq 1}} \sum_{\substack{p+h=k\\ p,h\geq 1\\ p:\mathrm{odd}}} \left((-1)^s\binom{p-1}{s-1}+(-1)^{p+r}\binom{p-1}{r-1}\right)\widetilde{\zeta}(p)f^{\mathbf{o}}_h(\tau)X^{r-1}Y^{s-1}\\
&= \sum_{\substack{p+h=k\\ p,h\geq 1\\ p:\mathrm{odd}}} (Y^{h-1}-X^{h-1})(Y-X)^{p-1}\widetilde{\zeta}(p)f^{\mathbf{o}}_h(\tau),\\
Z^{\mathbf{oe}}_k(X,Y) &:= \sum_{\substack{r+s=k\\ r,s\geq 1}} \mathrm{Im}\ Z^{\mathbf{oe}}_{r,s} X^{r-1}Y^{s-1} = \sum_{\substack{r+s=k\\ r,s\geq 1\\ s:\mathrm{odd}}} \widetilde{\zeta}(s)f^{\mathbf{o}}_r(\tau)X^{r-1}Y^{s-1}.
\end{aligned}
$$

We note that the imaginary part of the R.H.S. of (12) is the coefficient of $X^{r-1}Y^{s-1}$ of $Z^{\mathbf{oe}}_k(X+Y,Y)+Z^{\mathbf{oo}}_k(X+Y,X)$. Since we have

$$
\begin{aligned}
&Z^{\mathbf{oe}}_k(X+Y,Y)+Z^{\mathbf{oo}}_k(X+Y,X)\\
&= \sum_{\substack{r+s=k\\ r,s\geq 1\\ s:\mathrm{odd}}} \widetilde{\zeta}(s)f^{\mathbf{o}}_r(\tau)(X+Y)^{r-1}Y^{s-1} + \sum_{\substack{r+s=k\\ r,s\geq 1\\ s:\mathrm{odd}}} (X^{r-1}-(X+Y)^{r-1})(-Y)^{s-1}\widetilde{\zeta}(s)f^{\mathbf{o}}_r(\tau)\\
&= \sum_{\substack{r+s=k\\ r,s\geq 1\\ s:\mathrm{odd}}} \left((X+Y)^{r-1}Y^{s-1}+(X^{r-1}-(X+Y)^{r-1})Y^{s-1}\right)\widetilde{\zeta}(s)f^{\mathbf{o}}_r(\tau)\\
&= \sum_{\substack{r+s=k\\ r,s\geq 1\\ s:\mathrm{odd}}} \widetilde{\zeta}(s)f^{\mathbf{o}}_r(\tau)X^{r-1}Y^{s-1},
\end{aligned}
$$

the assertion follows. Secondly, we prove the equalities of the real parts in Proposition 4. Again we use generating functions. Define

$$
\begin{aligned}
f^{\mathbf{eo}}_{r,s}(\tau) &= \sum_{\substack{m>m'>0\\ m\in\mathbf{ev},m'\in\mathbf{od}}} \varphi_r(m\tau)\varphi_s(m'\tau),\ \ f^{\mathbf{oe}}_{r,s}(\tau) = \sum_{\substack{m>m'>0\\ m\in\mathbf{od},m'\in\mathbf{ev}}} \varphi_r(m\tau)\varphi_s(m'\tau),\\
f^{\mathbf{oo}}_{r,s}(\tau) &= \sum_{\substack{m>m'>0\\ m\in\mathbf{od},m'\in\mathbf{od}}} \varphi_r(m\tau)\varphi_s(m'\tau),\\
\beta^{\mathbf{oo}}_{r,s}(\tau) &= \sum_{\substack{p+h=r+s\\ p,h\geq 1}} \left((-1)^s\binom{p-1}{s-1}+(-1)^{p+r}\binom{p-1}{r-1}\right)\beta_p f^{\mathbf{o}}_h(\tau),
\end{aligned}
$$

where $\beta_p = -B_p/2p!(=\widetilde{\zeta}(p), p : \text{even})$. Consider

$$
\begin{aligned}
Z^{\mathbf{o}}(X) &:= \sum_{r\geq 1} G_r^0 X^{r-1} + \alpha_4 \cdot X = \sum_{r\geq 1} f_r^{\mathbf{o}}(\tau) X^{r-1} + \alpha_4 \cdot X,\\
Z^{\mathbf{eo}}(X,Y) &:= \sum_{r,s\geq 1} \mathrm{Re}\ Z_{r,s}^{\mathbf{eo}} X^{r-1}Y^{s-1} = \sum_{r,s\geq 1} \Big(f_{r,s}^{\mathbf{eo}}(\tau) + \frac{1}{4}\varepsilon_{r,s}^{\mathbf{eo}}(\tau)\Big) X^{r-1}Y^{s-1},\\
Z^{\mathbf{oe}}(X,Y) &:= \sum_{r,s\geq 1} \mathrm{Re}\ Z_{r,s}^{\mathbf{oe}} X^{r-1}Y^{s-1} = \sum_{r,s\geq 1} \Big(f_{r,s}^{\mathbf{oe}}(\tau) + \beta_s f_r^{\mathbf{o}}(\tau) + \frac{1}{4}\varepsilon_{r,s}^{\mathbf{oe}}(\tau)\Big) X^{r-1}Y^{s-1},\\
Z^{\mathbf{oo}}(X,Y) &:= \sum_{r,s\geq 1} \mathrm{Re}\ Z_{r,s}^{\mathbf{oo}} X^{r-1}Y^{s-1} = \sum_{r,s\geq 1} \Big(f_{r,s}^{\mathbf{oo}}(\tau) + \beta_{r,s}^{\mathbf{oo}}(\tau) + \frac{1}{4}\varepsilon_{r,s}^{\mathbf{oo}}(\tau)\Big) X^{r-1}Y^{s-1},\\
P^{\mathbf{oe}}(X,Y) &:= \sum_{r,s\geq 1} \mathrm{Re}\ \left(f_r^{\mathbf{o}}(\tau)(\widetilde{\zeta}(s) + f_s^{\mathbf{e}}(\tau)) + \frac{1}{4}(\delta_{r,2}\overline{f}_s^{\mathbf{e}}(\tau) + \delta_{s,2}\overline{f}_r^{\mathbf{o}}(\tau))\right) X^{r-1}Y^{s-1},\\
P^{\mathbf{oo}}(X,Y) &:= \sum_{r,s\geq 1} \mathrm{Re}\ \left(f_r^{\mathbf{o}}(\tau) f_s^{\mathbf{o}}(\tau) + \frac{1}{4}(\delta_{r,2}\overline{f}_s^{\mathbf{o}}(\tau) + \delta_{s,2}\overline{f}_r^{\mathbf{o}}(\tau))\right) X^{r-1}Y^{s-1},\\
\beta^{\mathbf{oo}}(X,Y) &= \sum_{r,s\geq 1} \beta_{r,s}^{\mathbf{oo}}(\tau) X^{r-1}Y^{s-1}\\
&= \sum_{p,h\geq 1} \beta_p f_h^{\mathbf{o}}(\tau) \sum_{r+s=p+h} \left((-1)^s \binom{p-1}{s-1} + (-1)^{p+r}\binom{p-1}{r-1}\right) X^{r-1}Y^{s-1}\\
&= \sum_{p,h\geq 1} \beta_p f_h^{\mathbf{o}}(\tau)(X-Y)^{p-1}(Y^{h-1} - X^{h-1}) = \beta(X-Y)(f^{\mathbf{o}}(Y) - f^{\mathbf{o}}(X)), \qquad (20)
\end{aligned}
$$

where $\alpha_4 = -\alpha_3/2$. Then, it is sufficient to prove that

$$
P^{\mathbf{oe}}(X,Y) = Z^{\mathbf{oe}}(X,Y) + Z^{\mathbf{eo}}(Y,X) = Z^{\mathbf{oe}}(X+Y,Y) + Z^{\mathbf{oo}}(X+Y,X), \qquad (21)
$$

$$
P^{\mathbf{oo}}(X,Y) = Z^{\mathbf{oo}}(X,Y) + Z^{\mathbf{oo}}(Y,X) + \frac{Z^{\mathbf{o}}(X) - Z^{\mathbf{o}}(Y)}{X-Y} = Z^{\mathbf{eo}}(X+Y,Y) + Z^{\mathbf{eo}}(X+Y,X). \qquad (22)
$$

Now we check the equalities in (21) and (22). Write $\gamma(X)$ and $\gamma(X,Y)$ for the generating functions $\sum_{k\geq 1}\gamma_k X^{k-1}$ and $\sum_{r,s\geq 1}\gamma_{r,s}X^{r-1}Y^{s-1}$ associated with sequences $\{\gamma_k\}$ and $\{\gamma_{r,s}\}$

indexed by one and two integers, respectively. Then we have

$$
\begin{aligned}
\beta(X) &= \sum_{k\geq 1} \beta_k X^{k-1} = \frac{1}{2}\left(\frac{1}{X} - \frac{1}{e^X - 1}\right), \\
f^{\mathbf{o}}(X) &= \sum_{k\geq 1} f_k^{\mathbf{o}}(\tau) X^{k-1} = -\sum_{u>0} e^{-uX} \frac{q^u}{1-q^{2u}}, \\
f^{\mathbf{e}}(X) &= \sum_{k\geq 1} f_k^{\mathbf{e}}(\tau) X^{k-1} = -\sum_{u>0} e^{-uX} \frac{q^{2u}}{1-q^{2u}}, \\
\overline{f}^{\mathbf{o}}(X) &= \sum_{k\geq 1} \overline{f}_k^{\mathbf{o}}(\tau) X^{k-1} = \frac{1}{X}\left(\sum_{u>0} e^{-uX} \frac{2q^u}{(1-q^{2u})^2} + f^{\mathbf{o}}(X) - \overline{f}_0^{\mathbf{o}}(\tau)\right), \\
\overline{f}^{\mathbf{e}}(X) &= \sum_{k\geq 1} \overline{f}_k^{\mathbf{o}}(\tau) X^{k-1} = \frac{1}{X}\left(\sum_{u>0} e^{-uX} \frac{2q^{2u}}{(1-q^{2u})^2} - \overline{f}_0^{\mathbf{e}}(\tau)\right), \\
f^{\mathbf{eo}}(X,Y) &= \sum_{r,s\geq 1} f_{r,s}^{\mathbf{eo}}(\tau) X^{r-1} Y^{s-1} = \sum_{u,v>0} e^{-uX-vY} \sum_{\substack{m>m'>0 \\ m\in\mathbf{ev},\, m'\in\mathbf{od}}} q^{um+vm'} \\
&= \sum_{u,v>0} e^{-uX-vY} \frac{q^u}{1-q^{2u}} \frac{q^{u+v}}{1-q^{2(u+v)}}, \\
f^{\mathbf{oe}}(X,Y) &= \sum_{r,s\geq 1} f_{r,s}^{\mathbf{oe}}(\tau) X^{r-1} Y^{s-1} = \sum_{u,v>0} e^{-uX-vY} \frac{q^u}{1-q^{2u}} \frac{q^{2(u+v)}}{1-q^{2(u+v)}}, \\
f^{\mathbf{oo}}(X,Y) &= \sum_{r,s\geq 1} f_{r,s}^{\mathbf{oo}}(\tau) X^{r-1} Y^{s-1} = \sum_{u,v>0} e^{-uX-vY} \frac{q^{2u}}{1-q^{2u}} \frac{q^{u+v}}{1-q^{2(u+v)}},
\end{aligned}
$$

$$
\begin{aligned}
\varepsilon^{\mathbf{eo}}(X,Y) &= X\overline{f}^{\mathbf{o}}(Y) - Y\overline{f}^{\mathbf{o}}(Y) - \overline{f}_0^{\mathbf{o}}(\tau) + X\overline{f}^{\mathbf{e}}(X) + \overline{f}_0^{\mathbf{e}}(\tau) + \alpha_1, \\
\varepsilon^{\mathbf{oe}}(X,Y) &= X\overline{f}^{\mathbf{e}}(Y) - Y\overline{f}^{\mathbf{e}}(Y) - \overline{f}_0^{\mathbf{e}}(\tau) + X\overline{f}^{\mathbf{o}}(X) + \overline{f}_0^{\mathbf{o}}(\tau) + \alpha_2, \\
\varepsilon^{\mathbf{oo}}(X,Y) &= X\overline{f}^{\mathbf{o}}(Y) - Y\overline{f}^{\mathbf{o}}(Y) + X\overline{f}^{\mathbf{o}}(X) + 2f^{\mathbf{o}}(X) + \alpha_3.
\end{aligned}
$$

By the definitions (20), we find

$$
\begin{aligned}
Z^{\mathbf{o}}(X) &= f^{\mathbf{o}}(X) + \alpha_4 \cdot X, \\
Z^{\mathbf{eo}}(X,Y) &= f^{\mathbf{eo}}(X,Y) + \frac{1}{4}\varepsilon^{\mathbf{eo}}(X,Y), \\
Z^{\mathbf{oe}}(X,Y) &= f^{\mathbf{oe}}(X,Y) + f^{\mathbf{o}}(X)\beta(Y) + \frac{1}{4}\varepsilon^{\mathbf{oe}}(X,Y), \\
Z^{\mathbf{oo}}(X,Y) &= f^{\mathbf{oo}}(X,Y) + \beta^{\mathbf{oo}}(X,Y) + \frac{1}{4}\varepsilon^{\mathbf{oo}}(X,Y), \\
P^{\mathbf{oe}}(X,Y) &= f^{\mathbf{o}}(X)f^{\mathbf{e}}(Y) + f^{\mathbf{o}}(X)\beta(Y) + \frac{1}{4}(X\overline{f}^{\mathbf{e}}(Y) + Y\overline{f}^{\mathbf{o}}(X)), \\
P^{\mathbf{oo}}(X,Y) &= f^{\mathbf{o}}(X)f^{\mathbf{o}}(Y) + \frac{1}{4}(X\overline{f}^{\mathbf{o}}(Y) + Y\overline{f}^{\mathbf{o}}(X)).
\end{aligned}
$$

For the first equality in (22), we compute

$$
\begin{aligned}
&f^{\mathbf{oo}}(X,Y)+f^{\mathbf{oo}}(Y,X)\\
&=\sum_{u,v>0} e^{-uX-vY}\left(\frac{q^{2u}}{1-q^{2u}}+\frac{q^{2v}}{1-q^{2v}}\right)\frac{q^{u+v}}{1-q^{2(u+v)}}\\
&=\sum_{u,v>0} e^{-uX-vY}\left(\frac{q^{u}}{1-q^{2u}}\frac{q^{v}}{1-q^{2v}}-\frac{q^{u+v}}{1-q^{2(u+v)}}\right)\\
&=f^{\mathbf{o}}(X)f^{\mathbf{o}}(Y)-\sum_{w>u>0} e^{-(w-u)Y-uX}\frac{q^{w}}{1-q^{2w}}\quad (w=u+v)\\
&=f^{\mathbf{o}}(X)f^{\mathbf{o}}(Y)-\sum_{w>0}\frac{q^{w}}{1-q^{2w}}e^{-wY}\left(e^{Y-X}\frac{1-e^{(Y-X)(w-1)}}{1-e^{Y-X}}\right)\\
&=f^{\mathbf{o}}(X)f^{\mathbf{o}}(Y)+\frac{e^{Y}}{e^{X}-e^{Y}}f^{\mathbf{o}}(Y)-\frac{e^{X}}{e^{X}-e^{Y}}f^{\mathbf{o}}(X)\\
&=f^{\mathbf{o}}(X)f^{\mathbf{o}}(Y)-\frac{1}{2}(f^{\mathbf{o}}(X)+f^{\mathbf{o}}(Y))-\frac{1}{2}\coth\left(\frac{X-Y}{2}\right)(f^{\mathbf{o}}(X)-f^{\mathbf{o}}(Y)),
\end{aligned}
$$

$$
\begin{aligned}
\beta^{\mathbf{oo}}(X,Y)+\beta^{\mathbf{oo}}(Y,X)&=(\beta(Y-X)-\beta(X-Y))(f^{\mathbf{o}}(X)-f^{\mathbf{o}}(Y))\\
&=-\frac{f^{\mathbf{o}}(X)-f^{\mathbf{o}}(Y)}{X-Y}+\frac{1}{2}\coth\left(\frac{X-Y}{2}\right)(f^{\mathbf{o}}(X)-f^{\mathbf{o}}(Y)),\\
\varepsilon^{\mathbf{oo}}(X,Y)+\varepsilon^{\mathbf{oo}}(Y,X)&=X\overline{f}^{\mathbf{o}}(Y)+Y\overline{f}^{\mathbf{o}}(X)+2f^{\mathbf{o}}(X)+2f^{\mathbf{o}}(Y)+2\alpha_3.
\end{aligned}
$$

Combining these with $(Z^{\mathbf{o}}(X)-Z^{\mathbf{o}}(Y))/(X-Y)$, we have

$$
\begin{aligned}
&Z^{\mathbf{oo}}(X,Y)+Z^{\mathbf{oo}}(Y,X)+\frac{Z^{\mathbf{o}}(X)-Z^{\mathbf{o}}(Y)}{X-Y}\\
&=f^{\mathbf{o}}(X)f^{\mathbf{o}}(Y)-\frac{1}{2}(f^{\mathbf{o}}(X)+f^{\mathbf{o}}(Y))-\frac{1}{2}\coth\left(\frac{X-Y}{2}\right)(f^{\mathbf{o}}(X)-f^{\mathbf{o}}(Y))\\
&-\frac{f^{\mathbf{o}}(X)-f^{\mathbf{o}}(Y)}{X-Y}+\frac{1}{2}\coth\left(\frac{X-Y}{2}\right)(f^{\mathbf{o}}(X)-f^{\mathbf{o}}(Y))\\
&+\frac{1}{4}\left(X\overline{f}^{\mathbf{o}}(Y)+Y\overline{f}^{\mathbf{o}}(X)+2f^{\mathbf{o}}(X)+2f^{\mathbf{o}}(Y)+2\alpha_3\right)+\frac{f^{\mathbf{o}}(X)-f^{\mathbf{o}}(Y)}{X-Y}+\alpha_4\\
&=f^{\mathbf{o}}(X)f^{\mathbf{o}}(Y)+\frac{1}{4}(X\overline{f}^{\mathbf{o}}(Y)+Y\overline{f}^{\mathbf{o}}(X)).
\end{aligned}
$$

For the second equality of (22), we proceed as follows:

$$
\begin{aligned}
&f^{\mathbf{eo}}(X+Y,X)+f^{\mathbf{eo}}(X+Y,Y)\\
&=\sum_{u,v>0}\left(e^{-(u+v)X-uY}+e^{-uX-(u+v)Y}\right)\frac{q^u}{1-q^{2u}}\frac{q^{u+v}}{1-q^{2(u+v)}}\\
&=\left(\sum_{w>u>0}+\sum_{u>w>0}\right)e^{-uX-wY}\frac{q^u}{1-q^{2u}}\frac{q^w}{1-q^{2w}}\\
&=\left(\sum_{w,u>0}-\sum_{w=u>0}\right)e^{-uX-wY}\frac{q^u}{1-q^{2u}}\frac{q^w}{1-q^{2w}}\\
&=f^{\mathbf{o}}(X)f^{\mathbf{o}}(Y)-\sum_{u>0}e^{-u(X+Y)}\frac{q^{2u}}{(1-q^{2u})^2}\\
&=f^{\mathbf{o}}(X)f^{\mathbf{o}}(Y)-\frac{1}{2}\overline{f}^{\mathbf{e}}_0(\tau)-\frac{1}{2}(X+Y)\overline{f}^{\mathbf{e}}(X+Y),\\
&\varepsilon^{\mathbf{eo}}(X+Y,X)+\varepsilon^{\mathbf{eo}}(X+Y,Y)\\
&=Y\overline{f}^{\mathbf{o}}(X)+X\overline{f}^{\mathbf{o}}(Y)+2(X+Y)\overline{f}^{\mathbf{e}}(X+Y)+2(\alpha_1-\overline{f}^{\mathbf{o}}_0(\tau)+\overline{f}^{\mathbf{e}}_0(\tau)).
\end{aligned}
$$

Summing these up, we have

$$
\begin{aligned}
&Z^{\mathbf{eo}}(X+Y,Y)+Z^{\mathbf{eo}}(X+Y,X)\\
&=f^{\mathbf{o}}(X)f^{\mathbf{o}}(Y)-\frac{1}{2}\overline{f}^{\mathbf{e}}_0(\tau)-\frac{1}{2}(X+Y)\overline{f}^{\mathbf{e}}(X+Y)\\
&+\frac{1}{4}\left(Y\overline{f}^{\mathbf{o}}(X)+X\overline{f}^{\mathbf{o}}(Y)+2(X+Y)\overline{f}^{\mathbf{e}}(X+Y)+2(\alpha_1-\overline{f}^{\mathbf{o}}_0(\tau)+\overline{f}^{\mathbf{e}}_0(\tau))\right)\\
&=f^{\mathbf{o}}(X)f^{\mathbf{o}}(Y)+\frac{1}{4}(X\overline{f}^{\mathbf{o}}(Y)+Y\overline{f}^{\mathbf{o}}(X)).
\end{aligned}
$$

For the first equality of (21), we compute

$$
\begin{aligned}
&f^{\mathbf{oe}}(X,Y)+f^{\mathbf{eo}}(Y,X)\\
&=\sum_{u,v>0}e^{-uX-vY}\left(\frac{q^v}{1-q^{2v}}+\frac{q^{2u+v}}{1-q^{2u}}\right)\frac{q^{u+v}}{1-q^{2(u+v)}}\\
&=\sum_{u,v>0}e^{-uX-vY}\frac{q^u}{1-q^{2u}}\frac{q^{2v}}{1-q^{2v}}=f^{\mathbf{o}}(X)f^{\mathbf{e}}(Y),\\
&\varepsilon^{\mathbf{oe}}(X,Y)+\varepsilon^{\mathbf{eo}}(Y,X)=X\overline{f}^{\mathbf{e}}(Y)+Y\overline{f}^{\mathbf{o}}(X)+\alpha_1+\alpha_2,
\end{aligned}
$$

to obtain

$$
Z^{\mathbf{oe}}(X,Y)+Z^{\mathbf{eo}}(Y,X)=f^{\mathbf{o}}(X)f^{\mathbf{e}}(Y)+f^{\mathbf{o}}(X)\beta(Y)+\frac{1}{4}(X\overline{f}^{\mathbf{e}}(Y)+Y\overline{f}^{\mathbf{o}}(X)).
$$

Finally, for the second equality of (21), we similarly compute

$$\begin{aligned}
&f^{\mathbf{oe}}(X+Y,Y)+f^{\mathbf{oo}}(X+Y,X)\\
&=\left(\sum_{w>u>0}+\sum_{u>w>0}\right)e^{-uX-vY}\frac{q^u}{1-q^{2u}}\frac{q^{2w}}{1-q^{2w}}\\
&=\left(\sum_{w,u>0}-\sum_{u=w>0}\right)e^{-uX-vY}\frac{q^u}{1-q^{2u}}\frac{q^{2w}}{1-q^{2w}}\\
&=f^{\mathbf{o}}(X)f^{\mathbf{e}}(Y)-\sum_{u>0}e^{-u(X+Y)}\frac{q^{3u}}{(1-q^{2u})^2}\\
&=f^{\mathbf{o}}(X)f^{\mathbf{e}}(Y)-\sum_{u>0}e^{-u(X+Y)}\left(\frac{q^u}{(1-q^{2u})^2}-\frac{q^u}{1-q^{2u}}\right)\\
&=f^{\mathbf{o}}(X)f^{\mathbf{e}}(Y)-\frac{1}{2}\left((X+Y)\overline{f}^{\mathbf{o}}(X+Y)-f^{\mathbf{o}}(X+Y)+\overline{f}^{\mathbf{o}}_0(\tau)\right)-f^{\mathbf{o}}(X+Y)\\
&=f^{\mathbf{o}}(X)f^{\mathbf{e}}(Y)-\frac{1}{2}f^{\mathbf{o}}(X+Y)-\frac{1}{2}(X+Y)\overline{f}^{\mathbf{o}}(X+Y)-\frac{1}{2}\overline{f}^{\mathbf{o}}_0(\tau),\\
&f^{\mathbf{o}}(X+Y)\beta(Y)+\beta^{\mathbf{oo}}(X+Y,X)=f^{\mathbf{o}}(X)\beta(Y),
\end{aligned}$$

$$\begin{aligned}
&\varepsilon^{\mathbf{oe}}(X+Y,Y)+\varepsilon^{\mathbf{oo}}(X+Y,X)\\
&=X\overline{f}^{\mathbf{e}}(Y)+Y\overline{f}^{\mathbf{o}}(X)+2(X+Y)\overline{f}^{\mathbf{o}}(X+Y)+2f^{\mathbf{o}}(X+Y)-\overline{f}^{\mathbf{e}}_0(\tau)+\overline{f}^{\mathbf{o}}_0(\tau)+\alpha_3+\alpha_2,
\end{aligned}$$

which give

$$\begin{aligned}
&Z^{\mathbf{oe}}(X+Y,Y)+Z^{\mathbf{oo}}(X+Y,X)\\
&=f^{\mathbf{o}}(X)f^{\mathbf{e}}(Y)-\frac{1}{2}f^{\mathbf{o}}(X+Y)-\frac{1}{2}(X+Y)\overline{f}^{\mathbf{o}}(X+Y)-\frac{1}{2}\overline{f}^{\mathbf{o}}_0(\tau)+f^{\mathbf{o}}(X)\beta(Y)+\\
&\frac{1}{4}\left(X\overline{f}^{\mathbf{e}}(Y)+Y\overline{f}^{\mathbf{o}}(X)+2(X+Y)\overline{f}^{\mathbf{o}}(X+Y)+2f^{\mathbf{o}}(X+Y)-\overline{f}^{\mathbf{e}}_0(\tau)+\overline{f}^{\mathbf{o}}_0(\tau)+\alpha_3+\alpha_2\right)\\
&=f^{\mathbf{o}}(X)f^{\mathbf{e}}(Y)+f^{\mathbf{o}}(X)\beta(Y)+\frac{1}{4}(X\overline{f}^{\mathbf{e}}(Y)+Y\overline{f}^{\mathbf{o}}(X)),
\end{aligned}$$

and we are done.

□

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